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Supergravity in $(2+1)$ dimensions from $(3+1)$-dimensional supergravity

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Abstract. In the context of the formalism proposed by Stelle–West and Grignani–Nardelli, it is shown that Chern–Simons supergravity can be consistently obtained as a dimensional reduction of $(3+1)$-dimensional supergravity, when written as a gauge theory of the Poincaré group. The dimensional reductions are consistent with the gauge symmetries, mapping $(3+1)$-dimensional Poincaré supergroup gauge transformations onto $(2+1)$-dimensional Poincaré supergroup ones.

1 Introduction

Supergravity in $(2+1)$ [1,2] and in $(3+1)$ [3,4] dimensions can be formulated as a gauge theory of the Poincaré superalgebra. The first-order formalism permits one to write the 3-dimensional supergravity as a Chern–Simons theory [5], for which $(2+1)$-dimensional supergravity is a good theoretical laboratory for the construction of a quantum theory [6]. Then it is interesting to find a link between supergravities in $(2+1)$ and in $(3+1)$ dimensions.

The action for supergravity in $(2+1)$ dimensions, $S = \int \left(\varepsilon_{abc}R^{ab}e^{c} + 4\overline{\psi}D\psi\right)$, with ψ a two component Majorana spinor, is invariant under Lorentz rotations, Poincaré translations and supersymmetry transformations. The dreibein e_{μ}^{a}, the spin connection ω_{μ}^{ab} and the gravitino ψ_{μ}^{a} transform as components of a connection for the super Poincaré group. This means that the supersymmetry algebra implied by the corresponding supersymmetry transformations is the super Poincaré algebra.

$(3+1)$-dimensional supergravity invariant under the Poincaré supergroup is based on the supersymmetric extension of the Stelle–West–Grignani–Nardelli formalism (SWGN) [3, 7, 10]. The fundamental idea of the formalism is founded on the definition [3, 7, 9] of the vierbein V^A and the gravitino Ψ, which involves the Goldstone fields ξ^A, χ. In the supersymmetric extension of the SWGN formalism:
(i) the vierbein V^A is not identified with the component e^A of the gauge potential along the translation generators, but is given by

$$V^{A} = D\zeta^{A} + e^{A} + \mathrm{i}\left(2\overline{\psi} + D\overline{\chi}\right)\gamma^{A}\chi; \qquad (1)$$

(ii) the gravitino field is not identified with the component ψ of the gauge potential along the supersymmetry generator, but is given by

$$\overline{\Psi} = \overline{\psi} + D\overline{\chi},$$

where $D\zeta^{A} = \mathrm{d}\zeta^{A} + \omega^{AB}\zeta_{B}$, $D\chi = \mathrm{d}\chi - \frac{1}{2}\omega^{AB}\gamma_{AB}$ and where ω^{AB} is the spin connection.

The purpose of the present work is to find the supersymmetric extension of the successful formalism of [10,11]. This means that, in the context of the procedure of [10,11], $(3+1)$-dimensional supergravity can be dimensionally reduced to Chern–Simons supergravity. This procedure can be used because both supergravity in $(2+1)$ [5] and supergravity in $(3+1)$ dimensions [3, 4] can be formulated as theories genuinely invariant under the Poincaré supergroup.

This paper is organized as follows: In Sect. 2, we shall review some aspects of the supersymmetric extension of the Stelle–West formalism and of supergravity as a gauge theory of the Poincaré supergroup. The dimensional reduction is carried out in Sect. 3 where the principal features of the dimensional reduction process are presented. Section 4 concludes the work with brief comments.

2 Supergravity invariant under the Poincaré group

In this section we shall review some aspects of the supersymmetric extension of the Stelle–West formalism and of supergravity as a gauge theory of the Poincaré group. The main point of this section is to display the differences in the invariances of the supergravity action when different definitions of a vierbein are used.

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2.1 Non-linear realization

The non-linear realizations can be studied by the general method developed in [12, 13]. Following these references, we consider a Lie (super)group G and a subgroup H.

Let us call $\{\mathbf{V}_i\}_{i=1}^{n-d}$ the generators of H. We assume that the remaining generators $\{\mathbf{A}_l\}_{l=1}^{d}$ can be chosen so that they form a representation of H. In other words, the commutator $[\mathbf{V}_i, \mathbf{A}_l]$ should be a linear combination of the $\mathbf{A}_l$ alone. A group element $g \in G$ can be represented (uniquely) in the form

$$g = e^{\xi^l \mathbf{A}_l} h, \tag{2}$$

where h is an element of H. The ξ^l parametrize the coset space G/H. We do not specify here the parametrization of h. One can define the effect of a group element g_0 on the coset space by

$$g_0 g = g_0(e^{\xi^l \mathbf{A}_l} h) = e^{\xi'^l \mathbf{A}_l} h', \tag{3}$$

or

$$g_0 e^{\xi^l \mathbf{A}_l} = e^{\xi'^l \mathbf{A}_l} h_1, \tag{4}$$

where

$$\xi' = \xi'(g_0, \xi), \tag{5}$$

$$h_1 = h' h^{-1}, \tag{6}$$

$$h_1 = h_1(g_0, \xi). \tag{7}$$

If $g_0 - 1$ is infinitesimal, (4) implies

$$e^{-\xi^l \mathbf{A}_l} (g_0 - 1) e^{\xi^l \mathbf{A}_l} - e^{-\xi^l \mathbf{A}_l} \delta e^{\xi^l \mathbf{A}_l} = h_1 - 1. \tag{8}$$

The right-hand side of (8) is a generator of H.

Let us first consider the case in which $g_0 = h_0 \in H$. Then (4) gives

$$e^{\xi'^l \mathbf{A}_l} = h_0 e^{\xi^l \mathbf{A}_l} h_0^{-1}. \tag{9}$$

Since the A^l form a representation of H, this implies

$$h_1 = h_0; \quad h' = h_0 h. \tag{10}$$

The transformation from ξ to ξ' given by (9) is linear. On the other hand, consider now

$$g_0 = e^{\xi_0^l \mathbf{A}_l}. \tag{11}$$

In this case (4) becomes

$$e^{\xi_0^l \mathbf{A}_l} e^{\xi^l \mathbf{A}_l} = e^{\xi'^l \mathbf{A}_l} h. \tag{12}$$

This is a non-linear inhomogeneous transformation on ξ. The infinitesimal form (8) becomes

$$e^{-\xi^l \mathbf{A}_l} \xi_0^i \mathbf{A}_i e^{\xi^j \mathbf{A}_j} - e^{-\xi^l \mathbf{A}_l} \delta e^{\xi^i \mathbf{A}_i} = h_1 - 1. \tag{13}$$

The left-hand side of this equation can be evaluated, using the algebra of the group. Since the results must be a generator of H, one must set equal to zero the coefficient of $\mathbf{A}_l$. In this way one finds an equation from which $\delta\xi^i$ can be calculated.

The construction of a Lagrangian invariant under coordinate-dependent group transformations requires the introduction of a set of gauge fields $a = a_\mu^i \mathbf{A}_i \mathrm{d}x^\mu$, $\rho = \rho_\mu^i \mathbf{V}_i \mathrm{d}x^\mu$, $p = p_\mu^l \mathbf{A}_l \mathrm{d}x^\mu$, $v = v_\mu^i \mathbf{V}_i \mathrm{d}x^\mu$, associated respectively with the generators V_i and A_l. Hence $\rho + a$ is the usual linear connection for the gauge group G, and the corresponding covariant derivative is given by

$$D = \mathrm{d} + f(\rho + a) \tag{14}$$

and its transformation law under $g \in G$ is

$$g : (\rho + a) \to (\rho' + a') = \left[g(\rho + a)g^{-1} - \frac{1}{f}(\mathrm{d}g)g^{-1} \right], \tag{15}$$

where f is a constant which, as it turns out, gives the strength of the universal coupling of the gauge fields to all other fields.

We now consider the Lie algebra valued differential form [12]

$$e^{-\xi^l \mathbf{A}_l} [\mathrm{d} + f(\rho + a)] e^{\xi^l \mathbf{A}_l} = p + v. \tag{16}$$

The transformation laws for the forms $p(\xi, \mathrm{d}\xi)$ and $v(\xi, \mathrm{d}\xi)$ are easily obtained. In fact, using (11) and (12) one finds [14]

$$p' = h_1 p (h_1)^{-1}, \tag{17}$$

$$v' = h_1 v (h_1)^{-1} + h_1 d (h_1)^{-1}. \tag{18}$$

Equation (17) shows that the differential forms $p(\xi, \mathrm{d}\xi)$ are transformed linearly by a group element of the form (11). The transformation law is the same as by an element of H, except that now this group element $h_1(\xi_0, \xi)$ is a function of the variable ξ. Therefore any expression constructed with $p(\xi, \mathrm{d}\xi)$ which is invariant under the subgroup H will be automatically invariant under the entire group G, the elements of H operating linearly on ξ, the remaining elements non-linearly.

2.2 Supersymmetric Stelle–West formalism

The basic idea of the Stelle–West formalism is founded on the non-linear realizations in anti-de Sitter space [7]. The supersymmetric extension of this formalism [4] is based in the non-linear realizations of supersymmetry in anti-de Sitter space [14]. The formalism considers as G the graded Lie algebra

$$\begin{aligned}
[P_A, P_B] &= -\mathrm{i} m^2 J_{AB}, \\
[J_{AB}, P_C] &= \mathrm{i}\left(\eta_{AC} P_B - \eta_{BC} P_A\right), \\
[J_{AB}, J_{CD}] &= \mathrm{i}(\eta_{AC} J_{BD} - \eta_{BC} J_{AD} \\
&\quad + \eta_{BD} J_{AC} - \eta_{AD} J_{BC}), \\
[J_{AB}, Q_\alpha] &= \mathrm{i}\left(\gamma_{AB}\right)_{\alpha\beta} Q_\beta,
\end{aligned}$$

$$[P_A, Q_\alpha] = -\frac{i}{2} m(\gamma_A)_{\alpha\beta} Q_\beta, \tag{19}$$
$$\left[Q_\alpha, \overline{Q}_\beta\right] = -2\left(\gamma^A\right)_{\alpha\beta} P_A - 2m(\gamma^{AB})_{\alpha\beta} J_{AB},$$

having as generators Q_α, P_A and M_{AB}. It has as a subalgebra H that of the de Sitter group $SO(3,2)$ with generators P_A and M_{AB}. This, in turn, has as subalgebra L that of the Lorentz group $SO(3,1)$ with generators M_{ab}. An element of G can be uniquely represented in the form

$$g = e^{\overline{\chi}Q} h = e^{\overline{\chi}Q} e^{-i\xi^A P_A} l, \tag{20}$$

where $h \in H$ and $l \in L$. On can define the effect of a group element g_0 on the coset space G/H by

$$g_0 g = e^{\overline{\chi}' Q} h' = e^{\overline{\chi}' Q} e^{-i\xi'^A P_A} l' \tag{21}$$

or

$$g_0 e^{\overline{\chi}Q} = e^{\overline{\chi}' Q} h_1, \tag{22}$$
$$h_1 e^{-i\xi^A P_A} = e^{-i\xi'^A P_A} l_1, \tag{23}$$
$$l_1 l = l'. \tag{24}$$

Clearly $h_1 = h_1(g_0, \chi)$ and $l_1 = l_1(g_0, \chi, \xi)$.

If $g_0 - 1$ and $h_1 - 1$ are infinitesimals, (22) and (23) imply

$$e^{-\overline{\chi}Q}\left(g_0 - 1\right) e^{\overline{\chi}Q} - e^{-\overline{\chi}Q} \delta e^{\overline{\chi}Q} = h_1 - 1 \tag{25}$$
$$e^{i\xi^A \mathbf{P}_A}\left(h_1 - 1\right) e^{-i\xi^A \mathbf{P}_A} - e^{i\xi^A \mathbf{P}_A} \delta e^{-i\xi^A \mathbf{P}_A} = l_1 - 1. \tag{26}$$

We consider now the following cases. If $g_0 = l_0 \in L$ (22) and (23) give

$$e^{\overline{\chi}' Q} = l_0 e^{\overline{\chi}Q} l_0^{-1}, \tag{27}$$
$$h_1 = l_1 = l_0, \tag{28}$$
$$e^{-i\xi'^A P_A} = l_0 e^{-i\xi^A P_A} l_0^{-1}. \tag{29}$$

Both χ and ξ transform linearly. If, on the other hand, we know only that $g_0 = h_0 \in H$, in particular, if

$$g_0 = e^{-i\rho^A \mathbf{P}_A} \tag{30}$$

is a pseudo-translation, (22) gives

$$e^{\overline{\chi}' Q} = h_0 e^{\overline{\chi}Q} h_0^{-1}, \tag{31}$$
$$h_1 = h_0, \tag{32}$$

while (23) gives

$$h_0 e^{i\xi^A \mathbf{P}_a} = e^{-i\xi'^A P_A} l_1(h_0, \xi). \tag{33}$$

In this case χ transforms linearly, but the transformation law (33) of ξ under pseudo-translations is inhomogeneous and non-linear. Infinitesimally

$$e^{i\xi^A \mathbf{P}_A}\left(-i\rho^B \mathbf{P}_B\right) e^{-i\xi^A \mathbf{P}_A} - e^{i\xi^A \mathbf{P}_A} \delta e^{-i\xi^A \mathbf{P}_A} = l_1 - 1. \tag{34}$$

Finally, if

$$g_0 = e^{\overline{\varepsilon}Q} \tag{35}$$

is a supersymmetry transformation, one must use (22) and (23) as they stand. Observe, however, that (23) has the same form as (33) except for the fact that h_1 is a function of χ while h_0 is not. Therefore, the transformation law for ξ under a supersymmetry transformation has the same form as that under a de Sitter transformation but with parameters which depend in a well defined way on χ.

An explicit form for the transformation law of ξ^a under an infinitesimal AdS boost can be obtained from (34). The result is

$$\delta\xi^A = -\rho^A + \left(\frac{z\cosh z}{\sinh z} - 1\right)\left(\rho^A - \frac{\rho^B \xi_B \xi^A}{\xi^2}\right), \tag{36}$$

where $z = m\sqrt{(\xi^a \xi_a)} = m\xi$.

The transformation of ξ^A under an infinitesimal Lorentz transformation $l_0 = e^{\frac{i}{2}\kappa^{AB} J_{AB}}$ is

$$\delta\xi^A = \kappa^{AB} \xi_B, \tag{37}$$

and, under a local supersymmetry transformation (35), ξ^A transforms as

$$\begin{aligned}\delta\xi^A &= -i\left(1 + \frac{i}{6} m\overline{\chi}\chi\right)\overline{\varepsilon}\gamma^A \chi \\ &+ i\left(\frac{z\cosh z}{\sinh z} - 1\right)\left(\delta^A_B - \frac{\xi_B \xi^A}{\xi^2}\right)\left(1 + \frac{i}{6} m\overline{\chi}\chi\right)\overline{\varepsilon}\gamma^B \chi \\ &- 2im\left(1 + \frac{i}{6} m\overline{\chi}\chi\right)\overline{\varepsilon}\gamma^{AB}\chi\xi_B.\end{aligned} \tag{38}$$

Using (25) with $g_0 - 1 = \overline{\varepsilon}Q$, one finds that

$$\delta\chi = \varepsilon - \frac{i}{6} m\left(5\overline{\chi}\chi + \overline{\chi}\Gamma_A \chi \Gamma^A\right)\varepsilon + \frac{1}{9} m^2 \left(\overline{\chi}\chi\right)\varepsilon, \tag{39}$$

$$h_1 - 1 = \left(1 + \frac{i}{6} m\overline{\chi}\chi\right)\left(\overline{\varepsilon}\gamma^A \chi P_A + m\overline{\varepsilon}\gamma^{AB}\chi J_{AB}\right). \tag{40}$$

Working in first-order formalism, the gauge fields vierbein e^A, spin connection ω^{AB} and gravitino ψ are treated as independent. The key observation is that the (e^A, ω^{AB}, ψ), considered as a single entity, constitute a multiplet in the adjoint representation of the AdS supergroup. That is, we can write

$$A = \frac{1}{2} i\omega^{AB} J_{AB} - ie^A P_A + +\overline{\psi}Q, \tag{41}$$

where A is the gauge field of the AdS supergroup, P_A, J_{AB}, Q^α being the generators of the AdS boosts. Then, based on these, we can define the corresponding non-linear connections (V^a, W^{ab}, Ψ) from (16):

$$\frac{1}{2}\mathrm{i}W^{AB}\mathbf{J}_{AB}-\mathrm{i}V^{A}\mathbf{P}_{A}+\overline{\Psi}Q \tag{42}$$
$$=e^{\mathrm{i}\xi^{A}\mathbf{P}_{A}}e^{-\overline{\chi}Q}\times\left[d+\frac{1}{2}\mathrm{i}\omega^{AB}\mathbf{J}_{AB}-\mathrm{i}e^{A}\mathbf{P}_{A}+\overline{\psi}Q\right]e^{\overline{\chi}Q}e^{-\mathrm{i}\xi^{B}\mathbf{P}_{B}}.$$

If $G=OSp(1,4)$ and $H=SO(3,2)$, the gauge fields V^A form a square 4×4 matrix which is invertible and can be identified with the vierbein fields. In the same way we have that W^{AB} is a connection and that $\overline{\Psi}$ can be identified with the Rarita–Schwinger field. From (42) one can obtain the fields V^A, W^{AB}, Ψ in terms of the fields e^A, ω^{AB}, ψ. The results are given in (81), (83) and (84) of [4].

The corresponding transformation laws for V^a, W^{ab}, Ψ can be obtained from (17) and (18). In fact, one can verify that, under the AdS supergroup, the non-linear connections transform as

$$\overline{\Psi}'Q=h_1\left(\overline{\Psi}Q\right)(h_1)^{-1}, \tag{43}$$
$$-\mathrm{i}V'^{a}\mathbf{P}_a=h_1\left(-\mathrm{i}V^{a}\mathbf{P}_a\right)(h_1)^{-1}, \tag{44}$$
$$\frac{1}{2}\mathrm{i}W'^{ab}\mathbf{J}_{ab}=h_1\left(\frac{1}{2}\mathrm{i}W^{ab}\mathbf{J}_{ab}\right)(h_1)^{-1}+h_1d(h_1)^{-1}. \tag{45}$$

The non-linearity of the transformation with respect to the elements of G/H means that the labels associated with the parts of the algebra of G which generate G/H are no longer available as symmetry indices. In other words, the symmetry has been spontaneously broken from G to H. An irreducible representation of G will, in general, have several irreducible pieces with respect to H. Since, in constructing invariant actions, one only needs index saturation with respect to the subgroup H, as far as the invariance is concerned it is possible to select a subset of non-linear fields with respect to G, which form irreducible multiplets with respect to H.

2.3 Supergravity invariant under the AdS group

Within the supersymmetric extension of the Stelle–West formalism, the action for supergravity with cosmological constant [15] can be rewritten as

$$S=\int\varepsilon_{abcd}\mathcal{R}^{ab}V^cV^d+4\overline{\Psi}\gamma_5\gamma_a\mathcal{D}\Psi V^a+2\alpha^2\varepsilon_{abcd}V^aV^bV^cV^d+3\alpha\varepsilon_{abcd}\overline{\Psi}\gamma^{ab}\Psi V^cV^d, \tag{46}$$

which is invariant under the supersymmetric extension of the AdS group. From such equations we can see that the vierbein V^a and the gravitino field transform homogeneously according to the representation of the AdS superalgebra, but with the non-linear group element $h_1\in H$.

The corresponding equations of motion are obtained by varying the action with respect to $\xi^a, \chi, e^a, \omega^{ab}, \psi$. The field equations corresponding to the variation of the action with respect to ξ^a and χ are not independent equations. Following the same procedure as in [16], we find that equations of motion for supergravity genuinely invariant under Super AdS are

$$2\varepsilon_{abcd}\overline{\mathcal{R}}^{ab}V^c+4\overline{\Psi}\gamma_5\gamma_d\rho, \tag{47}$$
$$\overset{\wedge}{\mathcal{T}}{}^{,a}=0, \tag{48}$$
$$8\gamma_5\gamma_a\rho V^a-4\gamma_5\gamma_a\Psi\overset{\wedge}{\mathcal{T}}{}^{a}=0, \tag{49}$$

where

$$\overset{\wedge}{\mathcal{T}}{}^{a}=\mathcal{T}^{a}-\frac{\mathrm{i}}{2}\overline{\Psi}\gamma^a\Psi, \tag{50}$$
$$\overline{\mathcal{R}}^{ab}=\mathcal{R}^{ab}+4\alpha^2V^aV^b+\alpha\overline{\Psi}\gamma^{ab}\Psi=0, \tag{51}$$
$$\rho=\mathcal{D}\Psi-\mathrm{i}\alpha\gamma^a\Psi V^a. \tag{52}$$

2.4 Supergravity and the Poincaré group

Taking the limit $m\to0$ in (24), (73), (75), (76), (81), (83) and (84) one can see that
(i) the superalgebra (19) takes the form of the superalgebra of Poincaré

$$\begin{aligned}
[P_A,P_B]&=0,\\
[J_{AB},P_C]&=\mathrm{i}\left(\eta_{AC}P_B-\eta_{BC}P_A\right),\\
[J_{AB},J_{CD}]&=\mathrm{i}(\eta_{AC}J_{BD}-\eta_{BC}J_{AD}\\
&\quad+\eta_{BD}J_{AC}-\eta_{AD}J_{BC}),\\
[J_{AB},Q_\alpha]&=\mathrm{i}\left(\gamma_{AB}\right)_{\alpha\beta}Q_\beta,\\
[P_A,Q_\beta]&=0,\\
\left[Q_\alpha,\overline{Q}_\beta\right]&=-2\left(\gamma^A\right)_{\alpha\beta}P_A;
\end{aligned} \tag{53}$$

(ii) the transformation laws of ξ^A under an infinitesimal Poincaré translation, under an infinitesimal Lorentz transformation, and under a local supersymmetry transformation are given respectively by

$$\delta\xi^A=-\rho^A, \tag{54}$$
$$\delta\xi^A=\kappa^{AB}\xi_B, \tag{55}$$
$$\delta\xi^A=-\mathrm{i}\overline{\varepsilon}\gamma^A\chi, \tag{56}$$

where ρ^A, $\kappa^{AB}=-\kappa^{BA}$ and ε are the infinitesimal parameters corresponding to Poincaré translations, Lorentz rotations and supersymmetry, respectively;
(iii) the transformation laws of χ under an infinitesimal Poincaré translation, under an infinitesimal Lorentz transformation, and under a local supersymmetry transformation are given respectively by

$$\delta\chi=0, \tag{57}$$
$$\delta\chi=\frac{1}{2}\kappa^{AB}\gamma_{AB}\chi, \tag{58}$$
$$\delta\chi=-\varepsilon. \tag{59}$$

In this limit G is the Poincaré supergroup and $H = SO(3,1)$, and the fields vierbein V^A, the connection W^{AB}, and the Rarita–Schwinger field $\overline{\Psi}$ are given by

$$V^A = e^A + D\zeta^A + \mathrm{i}\left(2\overline{\psi} + D\overline{\chi}\right)\gamma^A\chi, \tag{60}$$

$$W^{AB} = \omega^{AB}, \tag{61}$$

$$\overline{\Psi} = \overline{\psi} + D\overline{\chi}, \tag{62}$$

where now

$$D = \mathrm{d} + \omega. \tag{63}$$

The corresponding components of the curvature two-form are now

$$\mathcal{T}^{\ A} = DV^A, \tag{64}$$

$$R^A_B = \mathrm{d}\omega^A_B + \omega^A_C\omega^C_B. \tag{65}$$

3 Supergravity in $(2+1)$ from supergravity in $(3+1)$

3.1 Supergravity in $(3+1)$

The limit $m \to 0$ of the action (46) is obviously the action for $N = 1$ supergravity in $(3+1)$ dimensions:

$$S = \int \varepsilon_{ABCD} R^{AB} V^C V^D + 4\overline{\Psi}\gamma_5\gamma_A D\Psi V^A, \tag{66}$$

which is genuinely invariant under the Poincaré group. In fact, $d = 3+1$ and $N = 1$ supergravity is based on the Poincaré supergroup, whose generators P_A, J_{AB}, Q^α satisfy the Lie superalgebra (53). Using this algebra and the general form for gauge transformations on A,

$$\delta A = -D\lambda = \mathrm{d}\lambda - [A, \lambda], \tag{67}$$

with

$$\lambda = \frac{1}{2}\mathrm{i}\kappa^{AB} J_{AB} - \mathrm{i}\rho^A P_A + \overline{\varepsilon}Q, \tag{68}$$

we see that e^A, ω^{AB}, and ψ, under local Lorentz rotations, transform as

$$\delta e^A = \kappa^A_B e^B; \quad \delta\omega^{AB} = -D\kappa^{AB}; \quad \delta\psi = -\frac{1}{2}\kappa^{AB}\gamma_{AB}\psi, \tag{69}$$

and under local Poincaré translations transform as

$$\delta e^A = D\rho^A; \quad \delta\omega^{AB} = 0; \quad \delta\psi = 0; \tag{70}$$

and under local supersymmetry transformations as

$$\delta e^A = -2\mathrm{i}\overline{\varepsilon}\gamma^A\psi; \quad \delta\omega^{AB} = 0; \quad \delta\psi = D\varepsilon. \tag{71}$$

This means that the vierbein V^A transforms, under the Poincaré supergroup, as

$$\delta V^A = \kappa^A_B V^B. \tag{72}$$

The space-time supertorsion $\hat{\mathcal{T}}^{\ A}$ is given by

$$\hat{\mathcal{T}}^{\ A} = \mathcal{T}^{\ A} - \frac{1}{2}\bar{\psi}\gamma^A\psi, \tag{73}$$

where

$$\mathcal{T}^{\ A} = DV^A. \tag{74}$$

It is straightforward to verify that the action (66) is invariant under (69), (70), (71), (54), (55), (56), (57), (58) and (59).

3.2 Dimensional reduction

The dimensional reduction process, as well as the notation, is similar to those used in [10,11]. Latin indices $a, b, c, \dots = 0, 1, 2$ and capital latin indices $A, B, C, \dots = 0, 1, 2, 3$ denote $(2+1)$ and $(3+1)$ internal (gauge) indices respectively. They are raised and lowered by the Minkowski metric

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \eta_{AB} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{75}$$

In the dimensional reduction the first three values of $A, B, C, \dots$ will denote the corresponding $(2+1)$ internal indices $a, b, c, \dots$, i.e. $A = (a, 3)$, $B = (b, 3)$, $C = (c, 3)$, ... We shall use the antisymmetric symbol ε^{ABCD} with $\varepsilon^{0123} = 1$ and in $(2+1)$ dimensions $\varepsilon^{abc} = \varepsilon^{abc3}$, so that $\varepsilon^{012} = 1$.

Following the procedure of [10] we carried out a dimensional reduction of the Poincaré generators of the $(3+1)$-dimensional theory and, correspondingly, of the space-time dimensions that, from the $(3+1)$-dimensional action (66) and the algebra (53), lead to the $(2+1)$-dimensional action. With such reductions from the (3+1) gauge transformations (69), (70), (71), (54), (55), (56), (57), (58) and (59), we shall obtain the corresponding gauge transformations in $(2+1)$ dimensions.

The dimensional reduction leading from the $(3+1)$-dimensional supergravity theory to $(2+1)$ Chern–Simons supergravity theory is given in Table 1 [10], where the γ's with multiple indices are antisymmetrized products of gamma matrices, which for d dimensions satisfy the relationship [20]

$$\gamma^{i_1 i_2 \cdots\cdots i_k} = \alpha\varepsilon^{i_1 i_2 \cdots\cdots i_d}\gamma_{i_{k+1}\cdots\cdots i_d}\gamma^{d+1}, \tag{76}$$

with

$$\alpha = \frac{1}{(d-k)}(-1)^{\frac{1}{2}k(k-1)+\frac{1}{2}d(d-1)}. \tag{77}$$

It is straightforward to verify that the $(3+1)$-gauge transformations (69), (70) and (71), with the identifications of this table of the dimensional reduction, are mapped onto

$$\delta\xi^a = \kappa^a{}_b\xi^b; \ \delta e^a = \kappa^a{}_b e^b; \quad \delta\omega^{ab} = -D\kappa^{ab};$$

$$\delta\psi = -\frac{1}{2}\kappa^{ab}\gamma_{ab}\psi; \tag{78}$$

Table 1. Dimensional reduction leading from the (3 + 1)-dimensional supergravity theory to (2 + 1) Chern–Simons supergravity theory

Dimensional reduction	
(3 + 1) dimensions	(2 + 1) dimensions
e^3	$\mathrm{d}x^3$
e^a	e^a
ω^{ab}	ω^{ab}
ω^{a3}	0
ζ^a	ζ^a
ζ^3	0
ρ^a	ρ^a
ρ^3	0
κ^{ab}	κ^{ab}
κ^{a3}	0
ψ	ψ
γ^{abc}	γ^{abc}
γ^3	0

$$\delta\xi^a = -\rho^a;\ \delta e^a = D\rho^a;\ \delta\omega^{ab} = 0;\ \delta\psi = 0; \tag{79}$$

$$\delta\xi^a = -\mathrm{i}\bar{\varepsilon}\gamma^a\chi;\ \delta e^a = -2\mathrm{i}\bar{\varepsilon}\gamma^a\psi;\ \delta\omega^{ab} = 0;$$
$$\delta\psi = D\varepsilon; \tag{80}$$

i.e. onto the correct (2 + 1)-dimensional gauge transformations. In particular, the quantities that are set to a constant in the table consistently have vanishing gauge transformations. In the same way we have

$$R^{AB} = \begin{pmatrix} R^{ab} & R^{a3} \\ R^{3b} & R^{33} \end{pmatrix} = \begin{pmatrix} R^{ab} & 0 \\ 0 & 0 \end{pmatrix}, \tag{81}$$

$$\omega^{AB} = \begin{pmatrix} \omega^{ab} & \omega^{a3} \\ \omega^{3b} & \omega^{33} \end{pmatrix} = \begin{pmatrix} \omega^{ab} & 0 \\ 0 & 0 \end{pmatrix}, \tag{82}$$

$$V^A = \begin{pmatrix} V^a \\ V^3 \end{pmatrix} = \begin{pmatrix} e^a + D\xi^a + \mathrm{i}(2\overline{\psi} + D\overline{\chi})\gamma^a\chi \\ \mathrm{d}x^3 \end{pmatrix}, \tag{83}$$

$$\Psi = \psi + D\chi, \tag{84}$$

where $D\xi^a = \mathrm{d}\xi^a + \omega^a_b\xi^b$; $D\chi = \mathrm{d}\chi - \frac{1}{2}\omega^{ab}\gamma_{ab}\chi$.

From (76) we see that, for $d = 4$ and $k = 3$,

$$\gamma^{ABC} = -\varepsilon^{ABCD}\gamma_D\gamma^5, \tag{85}$$

which allows one to write the action for (3 + 1)-dimensional supergravity in the form

$$S^{4\mathrm{D}} = \int \varepsilon_{ABCD}\left(R^{AB}V^CV^D + \frac{1}{3!}\overline{\Psi}\gamma^{ABC}V^D D\Psi\right). \tag{86}$$

By substituting the content of the table of dimensional reduction and (81) and (82) into the action (86), one gets

$$S^{4\mathrm{D}} = \int \left(2\varepsilon_{abc3}R^{ab}V^c + \frac{4}{3!}\varepsilon_{abc3}\overline{\Psi}\gamma^{abc}D\Psi\right)\mathrm{d}x^3. \tag{87}$$

Using (83) and (84) and the identity $\gamma_{ab} = -\mathrm{i}\varepsilon_{abc}\gamma^c$ we find that the first term is

$$2\varepsilon_{abc3}R^{ab}V^c\mathrm{d}x^3 = (2\varepsilon_{abc3}R^{ab}e^c + 2\varepsilon_{abc3}R^{ab}D\xi^c - 4R^{ab}\overline{\psi}\gamma_{ab}\chi - 2R^{ab}(D\overline{\chi})\gamma_{ab}\chi)\mathrm{d}x^3. \tag{88}$$

Using (76), $\gamma^{abc} = -\varepsilon^{abc}I$, and the identities $DD\chi = \frac{1}{2}R^{ab}\gamma_{ab}\chi$; $\overline{\chi}\gamma_{ab}\psi = -\overline{\psi}\gamma_{ab}\chi$, we find that the second term is

$$\frac{4}{3!}\varepsilon_{abc3}\overline{\Psi}\gamma^{abc}D\Psi\mathrm{d}x^3 = \left(\frac{4}{3!}\varepsilon_{abc3}\overline{\psi}\gamma^{abc}D\psi + 4D(\overline{\chi}D\psi) + 4R^{ab}\overline{\psi}\gamma_{ab}\chi + 2R^{ab}(D\overline{\chi})\gamma_{ab}\chi\right)\mathrm{d}x^3. \tag{89}$$

By substituting (88) and (89) in (87) we obtain

$$S^{4\mathrm{D}} = \int \left(2\varepsilon_{abc3}R^{ab}e^c + \frac{4}{3!}\varepsilon_{abc3}\overline{\psi}\gamma^{abc}D\psi. + 2\varepsilon_{abc3}R^{ab}D\xi^c + 4D(\overline{\chi}D\psi)\right)\mathrm{d}x^3.$$

Using the Bianchi identity $DR^{ab} = 0, \varepsilon_{abc}\varepsilon^{abc} = -3!$ and (76) with $d = 3$ and $k = 3$, we find that the action for (2 + 1)- supergravity is given by

$$S^{4D} \longrightarrow S^{3D} = \int \varepsilon_{abc}R^{ab}e^c + 4\overline{\psi}D\psi + \text{surface term}, \tag{90}$$

which proves that the dimensional reduction from (3 + 1)-dimensional supergravity to (2 + 1)-supergravity is possible.

4 Comments

We have shown that the successful formalism proposed in [10,11] can be extended to the supersymmetric case. That is, (3 + 1)-dimensional supergravity can be dimensionally reduced to supergravity in (2+1) dimensions following the method of [10,11].

Finally we can say that supergravity genuinely invariant under the Poincaré supergroup [3,4] is a natural context to connect, preserving the invariance under the Poincaré supergroup, such a theory with (2 + 1)-dimensional supergravity.

It is interesting to note that all the terms containing ξ^a, χ disappear from the action and that e^a, ψ can be interpreted as the space-time dreibein and gravitino, and yet the theory is invariant under the Poincaré supergroup, contrary to what happens in (3 + 1) dimensions. The absence of the ξ^a, χ variables of (90) and the interpretation of e^a, ω^{ab} and ψ as gauge fields makes of (90) an action that can be conceived as a Chern–Simons three form.

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